Thus, tubular composites of the type tested in the present work need further development with the aim of increasing specific stiffness in bending for the whole range of operating stresses.

LITERATURE CITED


DEPENDENCE OF THE ELASTIC MODULUS ON THE PHYSICAL PROPERTIES OF A MATERIAL

Yu. Ya. Leus and Kh. M. Khanukhov

The elastic moduli of materials are of theoretical and practical importance. This fundamental characteristic reflects the strength of the atomic interaction, and studies on it serve to define the changes under various conditions, as well as the dependence on other properties of the material. Such studies are also important because one needs precise values for the elastic modulus in designing instruments, machines, and structures, as well as in the development of new materials with given mechanical properties [1]. Information from many different branches of science and technology may be used here.

For example, in solid-state physics one draws a distinction between the adiabatic elastic modulus (when there is no heat transfer with the environment, in particular on instantaneous loading) and the isothermal value (when the temperature of the body is kept constant), which are related by an equation containing certain physical constants of the material (the temperature coefficient of bulk expansion \( \alpha_0 \), the specific heat \( c \), and the density \( \gamma \)) [2].

However, to calculate the value of one of these moduli, it is necessary to have the value of the other.

In thermodynamics there are certain relations between the elastic modulus and other physical properties of the material characterizing the bond strength.

For example, in the theory of specific heat there is the Mie-Gruneisen equation, which relates the pressure, volume, and temperature, and this gives the Gruneisen formula [3, 4] for \( \alpha_0 \), in which the elastic modulus is present inexplicitly. However, to use this formula in calculations one has to have the Gruneisen coefficient, which is determined by experiment.

In the theory of specific heat, the temperature dependence of the elastic moduli is related to the coefficient of expansion [3], and the similar behavior of the two leads to the ratio being constant. The empirical Porifven equation [3] is based on this, in which the elastic modulus is proportional to the ratio of the melting point to unit volume. The elastic modulus can be calculated from the Porifven equation by means of empirical constants for the material determined by experiment.

In [4], the Gruneisen equations of state have been used to derive semiempirical relationships for the adiabatic and isothermal bulk compression coefficients (the reciprocals of the bulk modulus), and experimental values have been given for certain materials, which agree well with the calculated ones. However, the use of these relationships involves the experimental determination of certain quantities appearing in them.

Therefore, we have empirical equations derived on a theoretical basis that provide an explicit or inexplicit relation between the elastic modulus and other physical properties of the material. It is difficult to use these relationships in practical calculations because of the need for experimental determination of certain parameters. Therefore, here we use the thermodynamic conservation laws for the states of materials [1, 5] together with particular laws for deformable solids [5] to derive a formula for the elastic modulus on the basis of certain physical constants of the material given in the reference literature.

There are analytical relationships describing the behavior of a material in different phase states, particularly ones relating the gaseous and solid states.

The density of the internal-stress energy in terms of variations for the deformed state of a solid can be calculated from [2]

\[ dU = TdS + \sigma d\sigma + \sigma_i d\sigma_i, \]  

(1)

where \( U \) is the internal energy of unit volume; \( T \), temperature; \( S \), entropy of unit volume; \( \sigma \), average stress; \( \varepsilon \), bulk strain; \( \sigma_i \), stress intensity; and \( \varepsilon_i \), deformation intensity.

The second term on the right in (1) characterizes the work involved in changing the volume, while the third characterizes the work in changing the shape.

As a material passes from the liquid state to the solid one at the temperature of the surroundings \( T_0 \), the isobaric–isothermal potential \( \Phi \) [2, 6] or the Gibbs potential varies by \( \Delta \Phi \), which is defined by the following formula [2, 6]:

\[ \Delta \Phi = \Delta U - T\Delta S - \sigma \Delta \varepsilon - \sigma_i \Delta \varepsilon_i. \]  

(2)

We differentiate (2) on the basis that the stress and strain intensities are independent of the bulk strain and temperature [2] to get

\[ \frac{\partial \Delta \Phi}{\partial \varepsilon} = \frac{\partial \Delta U}{\partial \varepsilon} - \frac{\partial T}{\partial \varepsilon} \Delta S - T \frac{\partial \Delta S}{\partial \varepsilon} - \frac{\partial \sigma}{\partial \varepsilon} \Delta \varepsilon - \sigma. \]  

(3)

We differentiate (1) neglecting quantities of the second order of smallness to get

\[ \frac{\partial \Delta U}{\partial \varepsilon} - T \frac{\partial \Delta S}{\partial \varepsilon} - \sigma = 0. \]  

(4)

We substitute (4) into (3) and use the fact that the differential of the isobaric–isothermal potential \( \Phi = 0 \) [6], which gives us

\[ - \frac{\partial T}{\partial \varepsilon} \Delta S - \frac{\partial \sigma}{\partial \varepsilon} \Delta \varepsilon = 0. \]  

(5)

We replace the partial derivatives by differentials to get

\[ d\sigma = \frac{\Delta S}{\Delta \varepsilon} dT. \]  

(6)

The second law of thermodynamics can be put in the form [6]

\[ \Delta S = \frac{\Delta Q}{T}, \]  

(7)

where \( Q \) is the heat released when unit volume of the body cools to the environmental temperature. If there is an infinitely small change in \( T \), this is given by the following formula [6]

\[ dQ = \gamma dT. \]  

(8)

The bulk deformation \( \Delta \varepsilon \) for an infinitely small change in \( T \) during the cooling of an isotropic body is given by the following [6]:

\[ d\varepsilon = \alpha_d dT. \]  

(9)

We substitute (7)-(9) into (6) to get

\[ d\sigma = - \frac{\gamma}{\alpha_d} \cdot \frac{dT}{T}. \]  

(10)

One assumes that the mean stress \( \sigma \) characterizes the state of stress in the body arising on transition from the liquid state to the solid one, i.e.,

\[ \sigma = \Delta \rho = \rho_0 \varepsilon. \]  

(11)
Relationships from the theory of elasticity \([6, 7]\) give the average stress as defined by Hooke's law:

\[
\sigma = \varepsilon K = \varepsilon \frac{E}{3(1-2\mu)} ,
\]

(12)

where \(p_0\) is the pressure in the solid at the environmental temperature, \(\mu\) is Poisson's ratio, and \(K = \frac{E}{3(1-2\mu)}\) is the bulk modulus, which does not vary up to very high pressures \([7]\), and it follows from (11) and (12) that this adequately defines the pressure in the initial state of the material, i.e.,

\[
p = K = \frac{E}{3(1-2\mu)} .
\]

(13)

The ratio \(\frac{c_T}{c_\circ}\) is independent of temperature and is formulated as Grüneisen's second rule \([4, 8]\) for solid crystalline and amorphous bodies.

We neglect the latent heat in the transition from the liquid state to the solid one because this is small by comparison with the heat released during cooling of the solid from the melting point to the environmental temperature, and we then integrate (10) with respect to temperature:

\[
\sigma = P = - \frac{c_T}{c_\circ} \int_{T_\circ}^{T_{\text{mp}}} \frac{dT}{T} = - \frac{c_T}{c_\circ} \int_{T_\circ}^{T_{\text{mp}}} \frac{dT}{T} = \frac{c_T}{c_\circ} \ln \frac{T_{\text{mp}}}{T_\circ} .
\]

(14)*

It follows from (11) and (12) that

\[
K = \frac{c_T}{c_\circ} \ln \frac{T_{\text{mp}}}{T_\circ} ,
\]

(15)

where \(T_{\text{mp}}\) and \(T_\circ\) are the melting point of the solid and the environmental temperature in °K.

Poisson's ratio \(\mu\) for a solid varies from 0.3 within the elastic limits to 0.5 at the yield point. We can take approximately that \(\mu = 0.33\) on the basis of the conservation of momentum for an isotropic and homogeneous body. Then \(K\), the bulk modulus, is numerically equal to the modulus of normal elasticity \(E\), i.e.,

\[
E = \frac{c_T}{c_\circ} \ln \frac{T_{\text{mp}}}{T_\circ} .
\]

(16)

Table 1 compares the experimental values of \(E\) with those calculated from (16). The table shows that the agreement between the experimental and calculated values of \(E\) for many materials is very satisfactory although there are some exceptions.

The elastic modulus defined by (16) is the adiabatic one, since above we have considered an adiabatic-isothermal process in which the potential (internal) energy of atomic interaction accumulates. Equation (16) enables one to calculate \(E\) at normal temperature with reasonable accuracy by reference to published data on thermophysical constants.

In conclusion we note that (16) may be of value not only in developing a new material but also in monitoring its state in a workpiece or a structure, as well as in analyzing and interpreting experimental data.

\underline{Derivation of the equation}

\[
p = \frac{c_T}{c_\circ} \ln \frac{T_{\text{mp}}}{T_\circ} ,
\]

The basic relation of thermodynamics \([2, 6]\) may be put in the following form:

\[
dU = TdS + pdV .
\]

(1)

We use differential relations between the derivatives of thermodynamic quantities \([6]\).

When the variables \(V\) and \(T\) are independent, we get the adiabatic differential equation:

\[
\left( \frac{\partial T}{\partial V} \right)_S = - \frac{T}{C_V} \left( \frac{\partial p}{\partial T} \right)_V .
\]

(2)

*Expression (14) can be derived from (1) also in two ways, which are given in the appendix.
### TABLE 1. Theoretical and Experimental Values of the Elastic Modulus for Certain Materials

<table>
<thead>
<tr>
<th>Material</th>
<th>( \gamma, \text{g/cm}^3 )</th>
<th>( \alpha_{\text{v, deg}} )</th>
<th>( \alpha_{\text{l, deg}^{-1}} )</th>
<th>( \tau_{\text{mp}, \text{K}} )</th>
<th>( 2 \times 10^{-4}, \text{MPa} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>2.7</td>
<td>0.22</td>
<td>23...27</td>
<td>931</td>
<td>7.1</td>
</tr>
<tr>
<td>Iron</td>
<td>7.85</td>
<td>0.11</td>
<td>11...13</td>
<td>1793</td>
<td>21.0</td>
</tr>
<tr>
<td>Steel</td>
<td>7.78</td>
<td>0.12</td>
<td>12</td>
<td>1573...1673</td>
<td>21.0</td>
</tr>
<tr>
<td>Zinc</td>
<td>3.3</td>
<td>0.1</td>
<td>26</td>
<td>692</td>
<td>9.6</td>
</tr>
<tr>
<td>Copper</td>
<td>8.85</td>
<td>0.09</td>
<td>13</td>
<td>1356</td>
<td>11.0...12.0</td>
</tr>
<tr>
<td>Nickel</td>
<td>8.85</td>
<td>0.09</td>
<td>13</td>
<td>1723</td>
<td>22.5</td>
</tr>
<tr>
<td>Lead</td>
<td>11.4</td>
<td>0.03</td>
<td>27...30</td>
<td>600</td>
<td>1.8</td>
</tr>
<tr>
<td>Glass</td>
<td>2.8</td>
<td>0.2</td>
<td>24...30</td>
<td>1673</td>
<td>16.0</td>
</tr>
<tr>
<td>Silicon</td>
<td>2.4</td>
<td>0.1</td>
<td>9</td>
<td>1673</td>
<td>16.0</td>
</tr>
</tbody>
</table>

Note. Testing temperature for all materials \( T = 273^\circ \text{K} \).

where \( c_V = T \left( \frac{\partial S}{\partial T} \right)_V \) is the specific heat at constant volume.

We transform (2) as follows:

\[
T \left( \frac{\partial S}{\partial T} \right)_T \left( \frac{\partial T}{\partial V} \right)_S = -T \left( \frac{\partial p}{\partial T} \right)_V.
\]  

We write (3) in terms of ordinary differentials:

\[
\frac{dS}{dT} \frac{dT}{dV} = -\frac{dp}{dT}.
\]  

It follows from the second law of thermodynamics that there is the following entropy change in the system in the inverse process:

\[
dS = \frac{dQ}{T};
\]

\[
dQ = Vc_VdT;
\]

\[
dV = Vc_adT,
\]

where \( c_a = 3\alpha \) is the temperature coefficient of bulk expansion, \( \alpha \) is the temperature coefficient of linear expansion, \( c_V \) is the volume specific heat, \( c \) and \( \gamma \) are the specific heat and density, and \( T \) is temperature in \( ^\circ \text{K} \).

We substitute (5)-(7) into (4) to get

\[
\frac{dQ}{TdV} = \frac{Vc_VdT}{TVc_adT} = -\frac{dp}{dT}.
\]  

Further transformations give

\[
dp = -\frac{c_V}{c_a} \frac{dT}{T}.
\]  

Integration of (9) with respect to \( T \) gives

\[
p = -\frac{c_V}{c_a} \ln \frac{T}{T_o} + \frac{T}{T_o}.
\]

2. With \( p \) and \( T \) as independent variables, the differential equation for the adiabatic takes the form

\[
\left( \frac{dT}{dp} \right)_s = -\frac{T}{c_p} \left( \frac{dV}{dT} \right)_p.
\]  

where \( c_p = T \left( \frac{\partial S}{\partial T} \right)_p \) is the specific heat at constant pressure \( p \).

We substitute the expression for \( c_p \) into (11) and convert to ordinary differentials to get

\[
T \frac{dS}{dp} = -T \frac{dV}{dT} ;
\]

\[
T \frac{dS}{dT} \frac{dT}{dp} = -T \frac{dV}{dT}.
\]
We substitute into (13) for

\[ dS = \frac{dQ}{T} = \frac{V \cdot c \cdot \gamma \cdot dT}{T} \quad (14) \]

which gives

\[ dV = V \cdot \alpha \cdot dT. \quad (15) \]

After transformation we get

\[ \frac{Vc \gamma dT}{T dp} = - \frac{V \alpha dT}{dT}. \quad (16) \]

We integrate (17) with respect to T to get

\[ p = \frac{c \gamma}{\alpha_0} \ln \frac{T_{mp}}{T_0}. \quad (18) \]

LITERATURE CITED

7. A. A. Il'yushin, Plasticity [in Russian], Gostekhizdat, Moscow (1948).